Lower Bounds for the Cop Number when the Robber is Fast

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Abstract

We consider a variant of the Cops and Robbers game where the robber can move t edges at a time, and show that in this variant, the cop number of a d-regular graph with girth larger than $2t+2$ is $\Omega(d^t)$. By the known upper bounds on the order of cages, this implies that the cop number of a connected n-vertex graph can be as large as $\Omega(n^{2/3})$ if $t \geq 2$, and $\Omega(n^{4/5})$ if $t \geq 4$. This improves the $\Omega(n^{\frac{t-3}{t-2}})$ lower bound of Frieze, Krivelevich, and Loh (Variations on Cops and Robbers, preprint, 2010) when $2 \leq t \leq 6$. We also conjecture a general upper bound $O(n^{t/t+1})$ for the cop number in this variant, generalizing Meyniel's conjecture.

1 Introduction

The game of Cops and Robbers, introduced by Nowakowski and Winkler [10] and independently by Quilliot [11], is a perfect information game played on a finite graph G. There are two players, a set of cops and a robber. Initially, the cops are placed onto vertices of their choice in G (where more than one cop can be placed at a vertex). Then the robber, being fully aware of the cops placement, positions herself in one of the vertices of G. Then the cops and the robber move in alternate rounds, with the cops moving first; however, players are permitted to remain stationary on their turn if they wish. The players use the edges of G to move from vertex to vertex. The cops win and the game ends if eventually a cop steps into the vertex currently occupied by the robber; otherwise, i.e., if the robber can elude the cops indefinitely, the robber wins. The parameter of interest is the *cop number* of G, which is defined as the minimum number of cops needed to ensure that the cops can win. We will assume that the graph G is simple and connected, because deleting multiple edges or loops does not affect the set of possible moves of the players, and the cop number of a disconnected graph obviously equals the sum of the cop numbers for each connected component.

For a survey of results on the cop number and related search parameters, see the survey by Hahn [7]. The most well known open question in this area is Meyniel's conjecture, published by Frankl in [5]. It states that for every graph G on n vertices, $O(\sqrt{n})$ cops are enough to win. This is asymptotically tight, i.e. for every n there exists an n-vertex graph with cop number $\Omega(\sqrt{n})$. The best upper bound found so far is $n2^{-(1-o(1))\sqrt{\log_2 n}}$ (see [6, 9, 12] for several proofs).

Here we consider the variant where in each move, the robber can take any path of length at most t from her current position, but she is not allowed to pass through a vertex occupied by a cop. The parameter t is called the *speed* of the robber. This variant was first considered by Fomin, Golovach, Kratochvíl, Nisse, and Suchan [4], who proved that computing the cop number is NP-hard for every t. Next, Frieze, Krivelevich, and Loh [6] showed that the cop number of an n-vertex graph can be as large as $\Omega(n^{\frac{t-3}{t-2}})$. They also asked whether there exist graphs with cop number $\omega(\sqrt{n})$ if $t = 2$. We give a positive answer to this question, proving the existence of graphs with cop number $\Omega(n^{2/3})$ for $t \geq 2$, and graphs with cop number $\Omega(n^{4/5})$ for $t \geq 4$. This improves their bound $\Omega(n^{\frac{t-3}{t-2}})$ when $2 \leq t \leq 6$. In Section 2 the lower bounds are proved, and in Section 3 a conjecture is proposed, predicting the asymptotic value of cop number in this general setting.

2 The lower bounds

Lemma 1. *Let t, d be positive integers with $t \leq d+1$, G be a $(d+1)$-regular graph with girth larger than $2t+2$ and $\alpha \in (0,1)$ be such that αd^t is an integer. Assume that the robber has speed t. Then the cop number of G is at least $\frac{\alpha(1-\alpha)d^{2t}}{2(t+2)(d+1)^t}$.*

Proof. Let us first define a few terms. A cop *controls* a vertex v if the cop is on v or on an adjacent vertex. A cop controls a path if it controls a vertex of the path. The cops control a path if there is a cop controlling it. A vertex r is *safe* if there exists a set S of vertices of size αd^t such that for each $s \in S$, there is an (r,s)-path of length t not controlled by the cops.

Assume that there are less than $\frac{\alpha(1-\alpha)d^{2t}}{2(t+2)(d+1)^t}$ cops in the game, and we will show that the robber can elude forever. We may assume that the cops all start in one vertex u, and the robber starts in a vertex v at distance $t+1$ from u. Let N be the set of vertices at distance t from v. Then by the girth condition, the cops control only one vertex from N, and since $|N| > d^t$, v is a safe vertex. Hence we just need to show that if the robber is in a safe vertex before the cops move, then she can move to a safe vertex after the cops move.

Assume that the robber is in a safe vertex r after her last move. Then by definition there exists a set S of vertices of size αd^t such that for each $s \in S$, there is an (r,s)-path of length t not controlled by the cops. Let U be the set of all vertices of these paths. Now, look at the situation after the cops move. There is no cop in U, thus the robber can move to any of the vertices in S in her turn, and it suffices to prove that there is a safe vertex in S. Note that the girth of the graph is larger than $2t+2$, so S is an independent set and no vertex outside U is adjacent to two distinct vertices of S. By an *escaping path* we mean a path of length t with its first vertex in S and second vertex not in U. Clearly every $s \in S$ is the starting vertex of exactly d^t escaping paths.

Claim. After the cops move, each cop controls at most $(t+2)(d+1)^t$ escaping paths.

Proof. We first prove that every vertex v is on at most $t(d+1)^{t-1} + (d+1)^t$ escaping paths, and if $v \notin S$ then v is on at most $t(d+1)^{t-1}$ escaping paths. Let $u_1 u_2 u_3 \dots u_{t+1}$ be an escaping path with $u_1 \in S$ and $u_2 \notin U$ such that v is its i-th vertex, i.e. $v = u_i$. Assume first that $i \neq 1$. Note that by definition we have $u_2 \notin U$, so u_1 is determined uniquely by u_2. There are (at most) $d+1$ choices for each of $u_{i-1}, \dots, u_2$,

and for each of $u_{i+1}, u_{i+2}, \dots, u_{t+1}$. Consequently, for each $2 \le i \le t+1$, v is the i-th vertex of at most $(d+1)^{t-1}$ escaping paths, so if $v \notin S$ then v is on at most $t(d+1)^{t-1}$ escaping paths. If $i = 1$ then $v \in S$ and there are at most $d+1$ choices for each of $u_2, u_3, \dots, u_{t+1}$, thus each $v \in S$ is the first vertex of at most $(d+1)^t$ escaping paths. This shows that v is on at most $t(d+1)^{t-1} + (d+1)^t$ escaping paths.

Since the robber was in a safe vertex before the cops move, no cop is in U at this moment. Hence, each cop can control at most one vertex from S, through which he can control at most $(d+1)^t + t(d+1)^{t-1}$ escaping paths. Through every other vertex he can control at most $t(d+1)^{t-1}$ escaping paths, and he controls $d+2$ vertices in total. Therefore he controls no more than $(d+1)^t+(d+2)t(d+1)^{t-1} \le (t+2)(d+1)^t$ escaping paths. □

Now, since there are less than $\frac{\alpha(1-\alpha)d^{2t}}{2(t+2)(d+1)^t}$ cops in the game, the cops control less than $\alpha(1-\alpha)d^{2t}/2$ of the escaping paths. Since S has αd^t vertices, and each path has two endpoints, there must be an $s \in S$ such that at most $(1-\alpha)d^t$ escaping paths starting from s are controlled. Consequently, there are αd^t uncontrolled escaping paths starting from s. Note that girth of G is larger than $2t$ so the other endpoints of these paths are distinct. Hence s is safe by definition and the robber moves to s. □

Corollary 1. *Let t be some fixed positive integer denoting the speed of the robber. If G is a d-regular graph (where $d \ge \max\{3, t\}$) with girth larger than $2t+2$, then the cop number of G is $\Omega(d^t)$.*

In order to use Corollary 1 to prove interesting lower bounds for the cop number, one should look at vertex-minimal regular graphs with large girth, known as *cages*. Here are two useful results on cages (see [3] for a survey):

Theorem 1 ([8])**.** *Let $g \ge 5$, and $d \ge 3$ be an odd prime power. Then there exists a d-regular graph of girth g with at most $2d^{1+\frac{3}{4}g-a}$ vertices, where $a = 4, 11/4, 7/2, 13/4$ for $g \equiv 0, 1, 2, 3 \pmod 4$, respectively.*

Theorem 2 ([2])**.** *Let $d \ge 3$ be a prime power. Then there exists a d-regular graph with girth 12 and at most $2d^5$ vertices.*

Theorem 3. *Let t be some fixed positive integer denoting the speed of the robber.*

(a) If $t \ge 2$ then for every n there exists an n-vertex graph with cop number $\Omega(n^{2/3})$.

(b) If $t \ge 4$ then for every n there exists an n-vertex graph with cop number $\Omega(n^{4/5})$.

Proof. (a) As the cop number will not decrease when the speed of the robber is increased, we just need to show the proposition for $t = 2$. Let $n \ge 54$ and d be the largest prime number such that $2d^3 \le n$. Since there exists a prime between d and $2d$, we have $n < 2(2d)^3$ so $d = \Theta(n^{1/3})$. By Theorem 1, there exists a d-regular graph H of girth 7 with at most $2d^3$ vertices. By Corollary 1 the cop number of H is $\Omega(d^2) = \Omega(n^{2/3})$. Let G be the graph formed by joining some vertex of H to an endpoint of a disjoint path with $n - |V(H)|$ vertices. It is easy to check that the cop number of G equals the cop number of H, which is $\Omega(n^{2/3})$.

(b) Again we just need to show the proposition for $t = 4$. Let $n \ge 486$ and d be the largest prime number such that $2d^5 \le n$. A similar argument shows that $d = \Theta(n^{1/5})$. By Theorem 2, there exists a d-regular graph H of girth 12 with at most $2d^5$ vertices. By Corollary 1 the cop number of H is $\Omega(d^4) = \Omega(n^{4/5})$. Let G be the graph formed by joining some vertex of H to an endpoint of a

disjoint path with $n - |V(H)|$ vertices. Then the cop number of G equals the cop number of H, which is $\Omega(n^{4/5})$. $\square$

3 Concluding remarks

Let $f_t(n)$ be the maximum possible cop number of a connected n-vertex graph assuming the robber has speed t. It is well-known (and also follows from Corollary 1 and Theorem 1 with $g = 5$) that $f_1(n) = \Omega(\sqrt{n})$. Meyniel conjectured that indeed $f_1(n) = \Theta(\sqrt{n})$. Frieze, Krivelevich, and Loh [6] showed that $f_t(n) = \Omega(n^{\frac{t-3}{t-2}})$ if $t \geq 3$. In this note we proved that $f_2(n) = \Omega(n^{2/3})$ and $f_4(n) = \Omega(n^{4/5})$. A natural question is that of the asymptotic behavior of $f_t(n)$.

Notice that if G is a d-regular graph with girth larger than $2t+2$, then Moore's bound gives $d = O(n^{1/t+1})$. Hence Corollary 1 cannot give a better bound than $f_t(n) = \Omega(n^{t/t+1})$. Generalizing Meyniel's conjecture, we conjecture that this is actually the asymptotic behavior of $f_t(n)$.

Conjecture. For every fixed t we have $f_t(n) = \Theta(n^{t/t+1})$.

Proving better upper bounds on the order of cages would imply that the conjecture is tight. Specifically, if for a fixed t, and infinitely many d, there exists a d-regular graph with girth larger than $2t+2$ on $O(d^{t+1})$ vertices, then $f_t(n) = \Omega(n^{t/t+1})$ (see Corollary 1).

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Addendum. Alon and the author [1] have recently extended the result of this note, and proved that $f_t(n) = \Omega(n^{t/t+1})$ for every fixed positive integer t.

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